Spin correlations and entanglement in partially magnetised ensembles of fermions

IOP Publishing
Journal of Physics B: Atomic, Molecular and Optical Physics
J. Phys. B: At. Mol. Opt. Phys. **49** (2016) 214002 (5pp)
doi:10.1088/0953-4075/49/21/214002

Spin correlations and entanglement in partially magnetised ensembles of fermions

G S Thekkadath[1], **Liang Jiang**[2] **and J H Thywissen**[1,3]

[1] Department of Physics and CQIQC, University of Toronto, M5S 1A7, Canada
[2] Department of Applied Physics, Yale University, New Haven, CT 06511, USA
[3] Canadian Institute for Advanced Research, Toronto, Ontario M5G 1Z8, Canada

E-mail: joseph.thywissen@utoronto.ca

Received 30 June 2016, revised 11 September 2016
Accepted for publication 14 September 2016
Published 10 October 2016

Abstract

We show that the singlet fraction p_s and total magnetisation (or polarisation) m can bound the minimum concurrence in an ensemble of spins. We identify $p_s > (1 - m^2)/2$ as a sufficient and tight condition for bipartite entanglement. Our proof makes no assumptions about the state of the system or symmetry of the particles, and can therefore be used as a witness for spin entanglement between fermions. We discuss the implications for recent experiments in which spin correlations were observed, and the prospect to study entanglement dynamics in the demagnetisation of a cold Fermi gas.

Keywords: ultracold atoms, magnetisation, entanglement, spin correlations, concurrence

(Some figures may appear in colour only in the online journal)

1. Introduction

Spin correlations have recently been observed in cold fermionic atoms as signatures of pairing, magnetism, and interaction strength [1–3]. It is interesting to ask if the observed correlations require pairwise entanglement. Since typical experimental samples contain thousands of atoms, full tomography is inaccessible; instead, one must find an entanglement witness based on a reduced set of measurements [4, 5]. A commonly explored approach has been to measure spin squeezing [6–12], however this approach is mainly limited to symmetric states or indistinguishable particles [6–8], and thus inapplicable to spin mixtures of fermions.

An alternate characterisation may come from the degree of polarisation (or magnetisation) m, and the spin-singlet fraction p_s of the ensemble (see section 2 for precise definitions). The singlet state plays a key role in the physics of ultracold fermions, since a singlet spin wave function is required for s-wave interactions, which are the only interactions not suppressed at low energy by a centrifugal barrier. For spin mixtures near a Feshbach resonance [13], the pairing fraction can be measured by an adiabatic rapid passage that projects interacting pairs onto molecular dimers [14–16]. This enables direct measurement of p_s in an ensemble. Singlet fraction is also proportional to the s-wave contact [2, 17–25], and singlet pairs in an optical superlattice can also be mapped or projected onto excited motional states [1, 26–31].

It is well known that $p_s > 1/2$ indicates pairwise entanglement in unpolarised ($m = 0$) ensembles [32]. The existence of a threshold is intuitive, since spin singlets are antisymmetric Bell states. Here we assume both m and p_s of an ensemble are measured, but make no assumptions about the form of the reduced two-body density operator $\hat{\rho}^{AB}$, which has 15 degrees of freedom.

We find that the concurrence $\mathcal{C}$ of the ensemble can be bounded:

$$\mathcal{C} \geqslant \max[p_s - \sqrt{(1 - p_s)^2 - m^2}, 0]. \tag{1}$$

Concurrence [33] is a non-negative number characterising the entanglement of two spin-1/2 systems, with a positive value implying entanglement. Thus equation (1) also delineates a bound on the singlet fraction of an arbitrary two-body state that is a *sufficient and tight condition for its entanglement*, namely

$$p_s > \frac{1 - m^2}{2}. \tag{2}$$

This 'singlet bound' is an extension of the Werner bound to partially polarised ($m > 0$) ensembles, where $p_s > 1/2$ is sufficient but is not a tight bound. Our proof makes no assumptions about the state and therefore is a condition for bipartite entanglement in any ensemble of spins. Thus, equation (1) can elucidate the nature of spin correlations in recent experiments with interacting ensembles of spin-half fermions, even in states far from equilibrium.

2. One- and two-body observables

The spin state of pairs in a spin-$1/2$ ensemble can be described using the antisymmetric singlet state $|s_0\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ and the symmetric triplet states $|t_0\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, $|t_1\rangle = |\uparrow\uparrow\rangle$, and $|t_{-1}\rangle = |\downarrow\downarrow\rangle$. These form an orthonormal set of basis states with well defined angular momentum quantum numbers $|S, S_z\rangle$. The most general state can be written

$$\begin{aligned}\hat{\rho}^{AB} &= p_s|s_0\rangle\langle s_0| + \sum_{i\in\{0,\pm1\}}(q_i|s_0\rangle\langle t_i| \\ &\quad + q_i^*|t_i\rangle\langle s_0|) + \sum_{i,j\in\{0,\pm1\}}(p_{ij}|t_i\rangle\langle t_j|),\end{aligned} \tag{3}$$

where the populations are normalised to $\mathrm{Tr}[\hat{\rho}^{AB}] = 1$, and p_s is the singlet fraction.

The magnetisation is $\boldsymbol{m} = (m_x, m_y, m_z) = \mathrm{Tr}[\hat{\boldsymbol{S}}\hat{\rho}^{AB}]$, where $\hat{\boldsymbol{S}} = (\hat{\boldsymbol{\sigma}}^A + \hat{\boldsymbol{\sigma}}^B)/2$ and $\hat{\boldsymbol{\sigma}}^{A,B}$ are the usual Pauli spin operators. The reduced one-body states (e.g., $\hat{\rho}_A = \mathrm{Tr}_B\hat{\rho}^{AB}$) are completely defined by a Bloch vector $\boldsymbol{v}$: $\hat{\rho}_{A,B} \to \hat{I}/2 + \boldsymbol{v}_{A,B}\cdot\hat{\boldsymbol{\sigma}}/2$, in which $\hat{I}$ is the identity operator. Since $\boldsymbol{m} = \frac{1}{2}\boldsymbol{v}_A + \frac{1}{2}\boldsymbol{v}_B$,

$$m^2 = \tfrac{1}{4}(v_A^2 + v_B^2 + 2v_Av_B\cos\beta), \tag{4}$$

where β is the angle between the two Bloch vectors, and $m = |\boldsymbol{m}|$.

In the problem we are considering, only the ensemble observables $\boldsymbol{m}$ and p_s are measured, not v_A, v_B, or β. One simple relation between m and p_s is given by the normalisation of probability:

$$p_s \leqslant 1 - m. \tag{5}$$

This can be shown by noting that $m_z = p_{11} - p_{-1-1}$ and $p_s + \sum_i p_{ii} = 1$, from which the singlet fraction is bounded by $p_s \leqslant 1 - |m_z| - p_{00}$. Since $|m_z| \leqslant m$, equation (5) follows.

3. Unentangled spins

Let us start by finding the singlet fraction of the separable state $\hat{\rho}^{AB} = \hat{\rho}^A\otimes\hat{\rho}^B$ where $\hat{\rho}^A$ and $\hat{\rho}^B$ can be different mixed states. Since we are seeking a relation between two rotationally invariant quantities, m and p_s, we are free to choose the coordinate system, and align $\hat{\rho}^A$ along the z axis in Bloch space. Then $\hat{\rho}^A = p_{A\uparrow}|\uparrow_A\rangle\langle\uparrow_A| + p_{A\downarrow}|\downarrow_A\rangle\langle\downarrow_A|$, whereas $\hat{\rho}^B$ remains arbitrary, and we write it as $\hat{\rho}^B = \sum_{ij}c_{ij}|i_B\rangle\langle j_B|$ where $i, j \in \{\uparrow, \downarrow\}$. The singlet fraction is

$$p_s = \tfrac{1}{2}p_{A\uparrow}c_{\downarrow\downarrow} + \tfrac{1}{2}p_{A\downarrow}c_{\uparrow\uparrow}. \tag{6}$$

In terms of the Bloch vectors, $p_{A\uparrow} = (1 + v_A)/2$ and $p_{A\downarrow} = (1 - v_A)/2$, whereas $c_{\downarrow\downarrow} = (1 - v_B\cos\beta)/2$ and $c_{\uparrow\uparrow} = (1 + v_B\cos\beta)/2$, thus

$$p_s = \tfrac{1}{4}(1 - v_Av_B\cos\beta). \tag{7}$$

Equation (7) has a simple interpretation for two pure states: when the first spin is along the $+z$ axis of the Bloch sphere, the antiparallel (spin-down) fraction of the second spin is equally split between singlets and triplet zeros [34].

For an ensemble of unentangled qubits, each of which is in the same unknown mixed state, Gisin noted that $p_s = (1 - m^2)/4$, and proposed measuring p_s as a more efficient determination of m than measuring $\boldsymbol{m}$ [35]. We recover this result from equation (7) with $v_A = v_B$ and $\beta = 0$. However for arbitrary $\boldsymbol{v}_A$ or $\boldsymbol{v}_B$, we can only bound p_s: eliminating β, with equation (4),

$$p_s = \tfrac{1}{2}\left(1 - m^2 + \tfrac{1}{4}(v_A^2 - 1 + v_B^2 - 1)\right) \leqslant \frac{1 - m^2}{2}, \tag{8}$$

where the inequality holds because $|v_A| \leqslant 1$ and $|v_B| \leqslant 1$. Note that separable pure ($v_A = v_B = 1$) states are examples of non-entangled states on the line $p_s = (1 - m^2)/2$, which demonstrates the tightness of equation (2). (If however magnetisation is known only along z, but the full magnetisation possibly lies along another direction, the singlet bound $p_s > (1 - m_z^2)/2$ is sufficient but no longer tight.)

We generalise the inequality in equation (8) to all non-entangled states by considering a mixture of separable states i.e. $\hat{\rho}^{AB} = \sum_k P_k\hat{\rho}_k^{AB}$ where P_k is the probability of $\hat{\rho}_k^{AB} = \hat{\rho}_k^A\otimes\hat{\rho}_k^B$. The singlet fraction p_{sk} of each $\hat{\rho}_k^{AB}$ is still bounded by equation (8), thus

$$\overline{p_s} = \sum_k P_kp_{sk} \leqslant \frac{1 - \sum_k P_km_k^2}{2} = \frac{1 - \overline{m^2}}{2} \tag{9}$$

since $\overline{m^2} = \sum_k P_km_k^2$. Hence if the two-body state is non-entangled i.e. $\hat{\rho}^{AB} = \sum_k P_k\hat{\rho}_k^A\otimes\hat{\rho}_k^B$, then $p_s \leqslant (1 - m^2)/2$ holds.

4. Concurrence of entangled states

The contrapositive must also be true: if $p_s > (1 - m^2)/2$, then $\hat{\rho}^{AB}$ is entangled. In fact, we find the concurrence [33] of $\hat{\rho}^{AB}$ can be bounded using p_s and m, without any additional assumptions.

First, we define a 'spun state' as the state $\hat{\rho}^{AB}$ averaged uniformly over local rotations about the z axis, $U_z(\theta) = U_z^A(\theta)\otimes U_z^B(\theta)$:

$$\langle\hat{\rho}^{AB}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\;\hat{U}_z^\dagger(\theta)\hat{\rho}^{AB}\hat{U}_z(\theta). \tag{10}$$

This transformation eliminates coherences between states in $\hat{\rho}^{AB}$ with different angular momentum quantum number S_z, since $\hat{U}_z(\theta) = \exp[\mathrm{i}\theta\hat{S}_z]$. Populations and coherence between

$|s_0\rangle$ and $|t_0\rangle$ are unaffected, leaving

$$\langle\hat{\rho}^{AB}\rangle = p_s|s_0\rangle\langle s_0| + q_0|s_0\rangle\langle t_0| + q_0^*|t_0\rangle\langle s_0| + \sum_{i\in\{0,\pm1\}}(p_{ii}|t_i\rangle\langle t_i|) \quad (11)$$

which now has only six degrees of freedom. Crucially, because rotation can be implemented using local operations and classical communication, the spun state is at most as entangled as the unspun state i.e. $\mathcal{C}(\hat{\rho}^{AB}) \geqslant \mathcal{C}(\langle\hat{\rho}^{AB}\rangle)$ [36].

Next, we constrain the state to have polarisation m. Choosing the z axis along the measured direction of $\boldsymbol{m}$,

$$\langle\hat{\rho}^{AB}\rangle = p_s|s_0\rangle\langle s_0| + a|t_0\rangle\langle t_0| + c\mathrm{e}^{\mathrm{i}\phi}|s_0\rangle\langle t_0| + c\mathrm{e}^{-\mathrm{i}\phi}|t_0\rangle\langle s_0| + \frac{b+m}{2}|t_1\rangle\langle t_1| + \frac{b-m}{2}|t_{-1}\rangle\langle t_{-1}|, \quad (12)$$

where the normalised populations are $p_s + a + b = 1$ and the coherence is $c = \eta\sqrt{ap_s}$ with $\eta \in [0, 1]$.

Finally, we explicitly compute the concurrence of $\langle\hat{\rho}^{AB}\rangle$. The eigenvalues of the matrix $R = \left[\sqrt{\langle\hat{\rho}^{AB}\rangle}\langle\widetilde{\hat{\rho}^{AB}}\rangle\sqrt{\langle\hat{\rho}^{AB}\rangle}\right]^{1/2}$, where $\langle\widetilde{\hat{\rho}^{AB}}\rangle = (\sigma_y\otimes\sigma_y)\langle\hat{\rho}^{AB}\rangle^*(\sigma_y\otimes\sigma_y)$ is the 'spin-flipped' state, are

$$\lambda_{1,2} = \frac{1}{\sqrt{2}}\left[a^2 + p_s^2 - 2c^2\cos 2\phi \pm\sqrt{(a^2 + p_s^2 - 2c^2\cos 2\phi)^2 - 4(c^2 - ap_s)^2}\right]^{1/2}, \quad (13)$$

$$\lambda_3 = \lambda_4 = \tfrac{1}{2}\sqrt{b^2 - m^2}. \quad (14)$$

The concurrence is then

$$\begin{aligned}\mathcal{C}(\langle\hat{\rho}^{AB}\rangle) &= \max[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4] \\ &= \max[0, \sqrt{(p_s - a)^2 + 4c^2\sin^2\phi} - \sqrt{b^2 - m^2}],\end{aligned} \quad (15)$$

which is nonzero when

$$p_s > \frac{1}{2}\left(\frac{1 - 2a - m^2}{1 - 2a + 2a\eta^2\sin^2\phi}\right). \quad (16)$$

Since $\mathcal{C}(\hat{\rho}^{AB}) \geqslant \mathcal{C}(\langle\hat{\rho}^{AB}\rangle)$, equation (16) provides a general bound on the singlet fraction for the entanglement of any $\hat{\rho}^{AB}$ with magnetisation m, triplet population a, and coherence c. With only the observables p_s and m, this yields a sufficient condition for entanglement:

$$p_s > \sup_{a,\eta,\phi}\left[\frac{1}{2}\frac{1 - 2a - m^2}{1 - 2a + 2a\eta\sin^2\phi}\right] = \frac{1 - m^2}{2}. \quad (17)$$

With $a = 0$ (which implies $c = 0$) and $p_s = (1 - m^2)/2$, we see that equation (12) reduces to separable pure states, which fulfils equation (8) and saturates the bound. Another special case is the Werner state $a = b/2 = (1 - p_s)/3$, $m = 0$, and $c = 0$, for which equation (16) becomes $p_s > 1/2$.

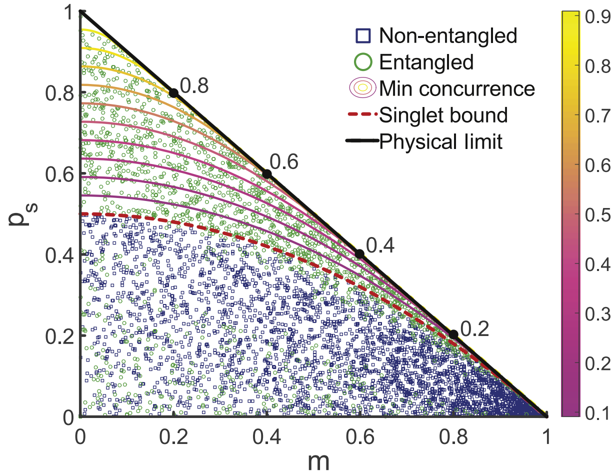

Figure 1. Each circle in singlet fraction p_s versus magnetisation m space is a randomly generated state $\langle\hat{\rho}^{AB}\rangle$ described by equation (11). Blue squares have $\mathcal{C} = 0$, while green circles have $\mathcal{C} > 0$. Note that there are blue squares immediately beneath the singlet bound equation (2) while there are none above, evidence that the bound is a tight and sufficient condition for entanglement. About the singlet bound, contour lines of $\mathcal{C}$ give the minimum concurrence (indicated by the colour scale) of a state with a given p_s and m. For a given $\mathcal{C}$, the line of minimum p_s follows equation (18) from $p_s = (1 + \mathcal{C})/2$ at $m = 0$ to $p_s = \mathcal{C}$ at $m = 1 - \mathcal{C}$. Several $\mathcal{C}$ values are given along physical limit (equation (5)).

The singlet bound found here (equation (2)) improves upon the generalised witness of [37], which when applied to $\langle\hat{\rho}^{AB}\rangle$ with $a = 0$, yields the sufficient condition $p_s \geqslant (1 + \sqrt{1 + 3\mathcal{C}^2})/3$.

The bound can be generalised to a threshold for finite concurrence, knowing only m and p_s, by noting that the minimum of equation (15) occurs when $a = 0$. Along with the constraint of a normalised probability, $p_s + b = 1$, this gives equation (1). Solving for p_s, this gives a tight and sufficient condition for $\hat{\rho}^{AB}$ having at least concurrence $\mathcal{C}$, namely

$$p_s \geqslant \frac{1 - \mathcal{C}^2 - m^2}{2(1 - \mathcal{C})} \quad \text{and} \quad m \leqslant 1 - \mathcal{C}, \quad (18)$$

where equation (2) is now found from the condition $\mathcal{C} > 0$. Equations (1) and (18) are the central results of our work.

We verify these relations by generating random mixed states that span the p_s and m space, and computing their concurrence. Each point in figure 1 corresponds to one of five thousand random spun mixed states. The blue squares have $\mathcal{C}(\langle\hat{\rho}^{AB}\rangle) = 0$ and are not entangled while green circles have $\mathcal{C}(\langle\hat{\rho}^{AB}\rangle) > 0$ and are entangled. All points lie within the physical limit, equation (5). The absence of non-entangled states above the singlet bound demonstrates that equation (2) is a sufficient condition for entanglement of $\hat{\rho}^{AB}$, while the existence of non-entangled states immediately beneath the bound demonstrates the tightness of the condition. Note that there are also entangled states below the singlet bound, as it is not a necessary condition for entanglement. Figure 1 also shows contour lines of minimum $\mathcal{C}$ determined from a larger set of random matrices. The locus of points with at least concurrence $\mathcal{C}$ or greater is bounded by equations (18) and (5).

5. Discussion

Several recent experimental works can be re-interpreted in light of our results. We will consider three measurements sensitive to p_s: mapping onto vibrational states in a superlattice, sweep-projection onto singlet dimers, and measuring the s-wave contact. We focus on experiments with fermions, even though our results apply to mixtures with any exchange statistics.

Controlled collisions in optical superlattices have been used both to create and to detect pairwise entanglement [1, 26–31]. However, when the effect of uncontrolled collisions are measured with the same technique, the efficiency of observing p_s may be hampered by a randomised choice of pairs, if a simple lattice is pairwise projected into the superlattice. For instance, Greif *et al* [1] find that in a dimerised lattice, the singlet fraction of fermion pairs is at least $p_s = 0.31$. This was an effective probe of spin correlations, but insufficient to prove entanglement by equation (2).

The association of atomic fermions into s-wave pairs also requires an initially singlet spin state. Thus efficiency $\mathcal{P}$ of association is a lower bound on p_s. For example, sweeping the magnetic field across a Feshbach resonance in experiments with unpolarised Fermi gases of ^{40}K and ^{6}Li, $\mathcal{P}$ as high as 85% is observed [16, 38, 39]. This surpasses the 50% upper limit discussed in [15, 40] which is also seen as an apparent limit in some experiments [14, 41]. We interpret this limit as $p_s = 0.5$, which is the maximum singlet fraction of a non-entangled state: experiments (and theoretical treatments) finding $\mathcal{P} \leqslant 0.5$ use separable states, whereas experiments observing $\mathcal{P} > 0.5$ allow multiple collisions to occur before or during the magnetic field ramp. In some conditions, these collisions have produced pairwise entanglement. From equation (1), we can infer that the concurrence was $\mathcal{C} \geqslant 0.7$ for $p_s \geqslant \mathcal{P} \approx 0.85$ in [16, 38, 39].

Pairwise-entangled states of an unpolarised Fermi gas are not surprising: in a weakly interacting Fermi s-wave superfluid, each spin-up fermion is (monogamously) entangled with a spin-down partner. However entanglement dynamics in a polarised gas is an active topic of discussion. Calculations of the s-wave contact $\mathcal{I}$ in a polarised Fermi gas [2, 42] have shown that $\mathcal{I} \propto 1 - m^2$ at high temperature, and $\mathcal{I} \propto 1 - m$ at low temperature. Since $\mathcal{I}$ reflects interaction strength, which in turn requires spin-singlet wave functions between fermions, this is similar to a study of p_s versus m. The conversion of $\mathcal{I}$ to an absolute value of p_s requires many-body theory and precise knowledge of density, temperature, and interaction strength. For this reason the spin correlations found by Bardon *et al* [2] using $\mathcal{I}$ and m, for instance, cannot easily be classified using the singlet bound. More clear would be to study demagnetisation dynamics using association efficiency $\mathcal{P}$. One would anticipate a temperature threshold, below which the gas evolves from a separable state to a pairwise-entangled state through random collisions.

In sum, we have established a sufficient and tight condition for bipartite entanglement between spin degrees of freedom in an arbitrary system of spins, without any assumption of equilibrium, population balance, or symmetry. We find that the concurrence can be bounded simply by the magnetisation and singlet fraction, through equation (1). This enables the distinction between classical spin correlations and necessarily quantum correlations in ensembles of ultracold fermions.

Acknowledgments

The authors thank D DeMille, A Dua, Xiwen Guan, B Sanders, C Simon, A M Steinberg, D Takahashi, E Taylor, S Trotzky, Shizhong Zhang, and Huaixiu Zheng for stimulating conversations, as well as K Heshami for assistance with the manuscript. LJ acknowledges support from ARO, AFOSR MURI, the Alfred P Sloan Foundation, and the Packard Foundation. JT acknowledges support from AFOSR under FA9550-13-1-0063, ARO under W911NF-15-1-0603, and NSERC.

References

[1] Greif D, Uehlinger T, Jotzu G, Tarruell L and Esslinger T 2013 *Science* **340** 1307–10
[2] Bardon A B *et al* 2014 *Science* **344** 722–4
[3] Hart R A, Duarte P M, Yang T L, Liu X, Paiva T, Khatami E, Scalettar R T, Trivedi N, Huse D A and Hulet R G 2015 *Nature* **519** 211–4
[4] Lewenstein M, Sanpera A, Ahufinger V, Damski B, Sen De A and Sen U 2007 *Adv. Phys.* **56** 243–379
[5] Guhne O and Toth G 2009 *Phys. Rep.* **474** 1–75
[6] Sørensen A, Duan L M, Cirac J I and Zoller P 2001 *Nature* **409** 63–6
[7] Wang X and Sanders B C 2003 *Phys. Rev.* A **68** 012101
[8] Korbicz J K, Cirac J I and Lewenstein M 2005 *Phys. Rev. Lett.* **95** 120502
[9] Kiesel N, Schmid C, Toth G, Solano E and Weinfurter H 2007 *Phys. Rev. Lett.* **98** 063604
[10] Tóth G, Knapp C, Gühne O and Briegel H J 2007 *Phys. Rev. Lett.* **99** 250405
[11] Estève J, Gross C, Weller A, Giovanazzi S and Oberthaler M 2008 *Nature* **455** 1216–9
[12] Baragiola B Q, Chase B A and Geremia J M 2010 *Phys. Rev.* A **81** 032104
[13] Chin C, Grimm R, Julienne P and Tiesinga E 2010 *Rev. Mod. Phys.* **82** 1225–86
[14] Strecker K E, Partridge G B and Hulet R G 2003 *Phys. Rev. Lett.* **91** 080406
[15] Pazy E, Vardi A and Band Y B 2004 *Phys. Rev. Lett.* **93** 120409
[16] Hodby E, Thompson S T, Regal C A, Greiner M, Wilson A C, Jin D S, Cornell E A and Wieman C E 2005 *Phys. Rev. Lett.* **94** 120402
[17] Partridge G B, Strecker K, Kamar R, Jack M and Hulet R 2005 *Phys. Rev. Lett.* **95** 020404
[18] Tan S 2008 *Ann. Phys.* **323** 2952–70
[19] Tan S 2008 *Ann. Phys.* **323** 2971–86
[20] Tan S 2008 *Ann. Phys.* **323** 2987–90
[21] Braaten E and Platter L 2008 *Phys. Rev. Lett.* **100** 205301

[22] Zhang S and Leggett A J 2009 *Phys. Rev.* A **79** 023601
[23] Werner F, Tarruell L and Castin Y 2009 *Eur. Phys. J.* B **68** 401–15
[24] Stewart J T, Gaebler J P, Drake T E and Jin D S 2010 *Phys. Rev. Lett.* **104** 235301
[25] Kuhnle E D, Hu H, Liu X J, Dyke P, Mark M, Drummond P D, Hannaford P and Vale C J 2010 *Phys. Rev. Lett.* **105** 070402
[26] Rey A, Gritsev V, Bloch I, Demler E and Lukin M 2007 *Phys. Rev. Lett.* **99** 140601
[27] Anderlini M, Lee P J, Brown B L, Sebby-Strabley J, Phillips W D and Porto J V 2007 *Nature* **448** 452–6
[28] Trotzky S, Cheinet P, Folling S, Feld M, Schnorrberger U, Rey A M, Polkovnikov A, Demler E A, Lukin M D and Bloch I 2008 *Science* **319** 295–9
[29] Trotzky S, Chen Y A, Schnorrberger U, Cheinet P and Bloch I 2010 *Phys. Rev. Lett.* **105** 265303
[30] Greif D, Tarruell L, Uehlinger T, Jördens R and Esslinger T 2011 *Phys. Rev. Lett.* **106** 145302
[31] Nascimbène S, Chen Y A, Atala M, Aidelsburger M, Trotzky S, Paredes B and Bloch I 2012 *Phys. Rev. Lett.* **108** 205301
[32] Werner R F 1989 *Phys. Rev.* A **40** 4277–81
[33] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245–8
[34] Aephraim S 2014 Private communication
[35] Gisin N 2001 *J. Mod. Opt.* **48** 1397–403
[36] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[37] Mintert F and Buchleitner A 2007 *Phys. Rev. Lett.* **98** 140505
[38] Cubizolles J, Bourdel T, Kokkelmans S J J M F, Shlyapnikov G V and Salomon C 2003 *Phys. Rev. Lett.* **91** 240401
[39] Greiner M, Regal C A and Jin D S 2003 *Nature* **426** 537–40
[40] Chwedeńczuk J, Goral K, Köhler T and Julienne P S 2004 *Phys. Rev. Lett.* **93** 260403
[41] Regal C A, Ticknor C, Bohn J L and Jin D S 2003 *Nature* **424** 47–50
[42] He W B, Chen Y Y, Zhang S and Guan X W 2016 Universal properties of Fermi gases in one dimension *Phys. Rev. A* **94** 031604