OPTIMIZATION OF QUANTUM REPEATER PROTOCOLS

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We consider a dynamic programming approach for the systematic optimization of standard quantum repeater protocols. For the specific optimization problem we choose to fix the distance and fidelity of a desired entangled pair (a fundamental resource for long-distance quantum communication) and attempt to minimize the time required to create such a pair. Taking advantage of the natural self-similar structure of quantum repeaters we can find a near-optimal solution to this problem without searching an exponentially growing space of solutions.

1. DESCRIPTION OF THE OPTIMIZATION PROBLEM

A standard problem in quantum communication is the fast generation of high fidelity entangled pairs at long distances. Quantum repeaters attempt to solve this problem in a time polynomial in the distance (overcoming fiber attenuation losses) by using a quantum memory and local quantum operations to purify and connect smaller distance pairs. In this paper we discuss progress towards optimization of two repeater protocols: the BDCZ protocol [1] and the CTSL protocol [2]. While the detailed results await a later publication [3], here we outline the necessary mathematical formulae for solving the optimization problem for the (simpler) BDCZ protocol, and consider specifically the role detector and readout efficiency play in the scaling properties of optimized repeater protocols, with specific examples.

We use computer-aided dynamic programming [4] to minimize the time to generate entangled pairs for specific distances and final fidelities. This approach takes advantage of the self-similar structure of the repeater protocol. In particular, we will find optimal approaches for the generation of shorter range pairs, which form the building blocks for generating longer range pairs. Thus, a computer may exhaustively search of a wide range of parameters for each recursive level of the procedure without incurring exponential overhead in the total number of repeater nodes.

We follow the notations of Ref. [2]. There are two types of entangled pairs used in the BDCZ protocol: type-A pairs, which are purified pairs used for the next level of the protocol, and type-B pairs, unpurified intermediate pairs used to purify type-A pairs. In either case, the process is repeated recursively to build long-range entanglement [1, 2], with the final A pair being the desired entangled state.

The control parameters to generate a purified type-A pair with distance n are

1. m – number of continuous successful entanglement pumping steps required to obtain a purified pair

2. k – during generation of type-B pair, the separating node at which two lower level of type-A pairs are generated. The two lower level type-A pairs span from node 1 to k, and from $k+1$ to n, respectively. The two type-A pairs have distance k and $n-k$ respectively.

Finally, our approach will be to create a table of solutions for given distances $n' \leq n$ and a range of fidelities $\{F\} = \{0.8, 0.805, \ldots, 0.99, 0.995\}$. These solutions, in turn, will be used as the building blocks for generating the next elements of the table for distance $n+1$. Proceeding inductively, the overall optimization procedure will take a time $O(n^2)$ and require recursive formulae for the generation time given a target fidelity. In the remainder of the paper, we consider these formulae, then analyze the dependence of the optimized protocol on detector efficiency and its performance at very long ranges.

2. RECURSIVE FORMULA FOR THE BDCZ PROTOCOL

For type-A pair, the average time for generating a pair of fidelity $F_A^{(n)}$ at distance n depends upon the success of m purification steps each using a type-B pair of fidelity $F_{B,i}^{(n)}$:

$$\begin{aligned} & T_A\left(n, F_{A,m}^{(n)}; m, F_{B,0}^{(n)}, F_{B,1}^{(n)}, F_{B,2}^{(n)}, \cdots, F_{B,m}^{(n)}\right) \\ = & \frac{1}{P_S\left(j\right)}\left[T_A\left(n, F_{A,m-1}^{(n)}; m-1, F_{B,0}^{(n)}, F_{B,1}^{(n)}, F_{B,2}^{(n)}, \cdots, F_{B,m-1}^{(n)}\right) + T_B^*\left(n, F_{B,m}^{(n)}\right) + n\ t_C\right] \\ = & \sum_{i=1}^{m}\left(\left[T_B^*\left(n, F_{B,i}^{(n)}\right) + nt_C\right]\prod_{j=i}^{m}\frac{1}{P_S\left(j\right)}\right) + T_B^*\left(n, F_{B,0}^{(n)}\right)\prod_{j=1}^{m}\frac{1}{P_S\left(j\right)} \end{aligned} \tag{1}$$

where t_C is the nearest-neighbor communication time and $P_S(j)$ is the success probability for the jth step of purification. Practically, we can set all the fidelities of type-B pairs to be equal, $F_{B,j}^{(n)} = F_B^{(n)}$; then the above formula reduces to:

$$\begin{aligned} & T_A\left(n, F_{A,m}^{(n)}; m, F_B^{(n)}\right) \\ = & \left(T_B^*\left(n, F_B^{(n)}\right) + nt_C\right)\sum_{i=1}^{m}\prod_{j=i}^{m}\frac{1}{P_S\left(j\right)} + T_B^*\left(n, F_B^{(n)}\right)\prod_{j=1}^{m}\frac{1}{P_S\left(j\right)} \end{aligned} \tag{2}$$

with fidelities of the purified pair after j consecutively successful purifications

$$F_{A,j}^{(n)} = \mathcal{P}\left(\left\{F_{A,j-1}^{(n)}, F_B^{(n)}\right\}, \eta, p\right) \tag{3}$$

where the function $\mathcal{P}\left(\{F_1, F_2\}, \eta, p\right)$ calculates the fidelity of the output pair of a successful purificaiton, which uses two entangled pairs (of fidelity F_1 and F_2) and imperfect local operations characterized by η and p. The success probability for the jth purification is

$$P_S\left(j\right) = P_S\left(\left\{F_{A,j-1}^{(n)}, F_B^{(n)}\right\}, \eta, p\right) \tag{4}$$

To find the optimal (minimum) time $T_A^*\left(n, F_A^{(n)}\right)$, we need to find the minimum in a four dimensional space spanned by $m, \underbrace{\left\{k, F_A^{(k)}, F_A^{(n-k)}\right\}}_{\text{B pair}}$.

We can further decompose the above optimization problem into two sub-problems by assuming that in order to generate type-A pair in minimal time, we generate type-B pair with the required fidelity also in minimal time.

Thus, for a type-B pair

$$T_B\left(n, F_B^{(n)}; k, F_A^{(k)}, F_A^{(n-k)}\right) = \max\left\{T_A^*\left(k, F_A^{(k)}\right), T_A^*\left(n-k, F_A^{(n-k)}\right)\right\} + (n-k)\, t_c \quad (5)$$

with fidelities of the unpurified pair

$$F_B^{(n)} = \mathcal{C}\left(\left\{F_A^{(k)}, F_A^{(n-k)}\right\}, \eta, p\right). \quad (6)$$

The function $\mathcal{C}\left(\{F_1, F_2, \cdots, F_h\}, \eta, p\right)$ calculates the fidelity of the entangled pair from connecting h entangled pairs with fidelities $\{F_1, F_2, \cdots, F_h\}$ by imperfect local operations characterized by η and p [2].

The optimization takes place by defining the optimal times

$$T_A^*\left(n, F_A^{(n)}\right) = \min_{m, F_B^{(n)}} T_A\left(n, F_A^{(n)}; m, F_B^{(n)}\right) \quad (7)$$

$$T_B^*\left(n, F_B^{(n)}\right) = \min_{k, F_A^{(k)}, F_A^{(n-k)}} T_B\left(n, F_B^{(n)}; k, F_A^{(k)}, F_A^{(n-k)}\right) \quad (8)$$

for each distance and fidelity pair.

The recursive relations end for $n = 2$ (nearest neighbor) type-B pairs. These are unpurified entangled pairs between neighboring repeater nodes, created by direct entanglement generation. The parameters for the entanglement generation process determines the relation between the average generation time, τ_e, and fidelity of the entangled pair between two neighboring stations, F_0:

$$F_0 = F_0\left(\tau_e\right) = \frac{1}{2}\left\{1 + \left[1 - \frac{L_0}{\tau_e c} e^{L_0/L_{att}}\right]^{2(1-\varepsilon)/\varepsilon}\right\}, \quad (9)$$

where the second equality assumes the specific entanglement generation scheme from Ref. [2].

3. DEPENDENCE ON PHOTON DETECTION EFFICIENCY

We now consider entanglement generation schemes which use single photons to generate entanglement between quantum memories two neighboring repeater stations. In reality, the collection and detection of single photons has finite efficiency, which not only slows down the entire entanglement generation process, but also reduces the fidelity of the generated entanglement pairs.

The overall efficiency is the product of all efficiencies throughout the optical path and detector, $\varepsilon = \varepsilon_c \varepsilon_d$, where ε_c and ε_d are the collection and detection efficiencies, respectively. For different values of efficiency ε, we plot fidelity v.s. time curves according to $F_0\left(\tau_e\right)$ relation (Eq.(9)) in Fig. 1a.

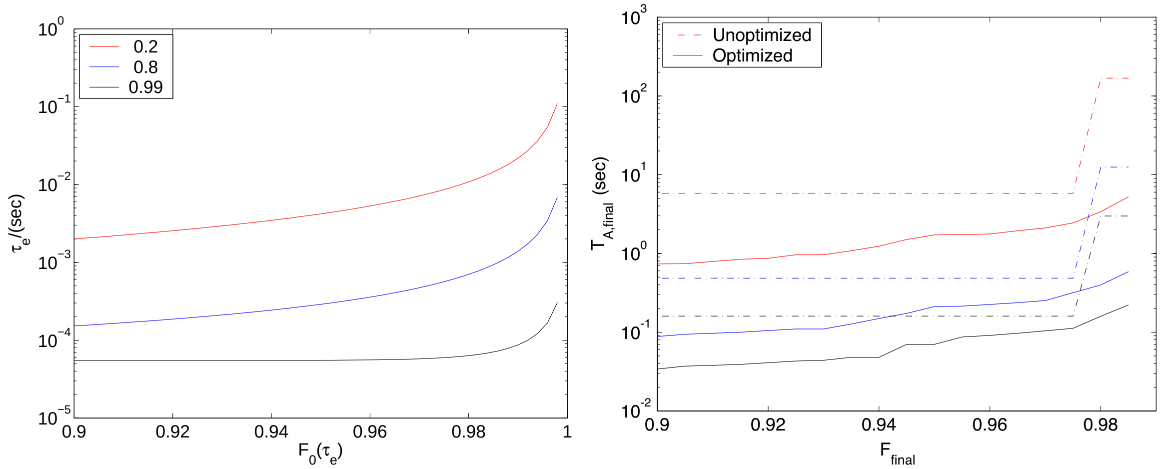

Figure 1: (a) Time vs. fidelity for nearest neighbor entanglement generation at efficiencies $\varepsilon = 0.2,\ 0.8,\ 0.99$ (from top to bottom). The distance between neighboring stations is $L_0 = 10km$, and fiber attenuation length is $L_{att} = 20km$. (b) Unoptimized (dashed) and optimized time (solid) vs. final fidelity for distance $L = 1280km$ for the BDCZ protocol, with efficiencies $\varepsilon = 0.2,\ 0.8,\ 0.99$ (from top to bottom).

In Fig. 1b, we plot the optimized time versus final fidelity at a distance $L = 1280km$. From the plot we can infer that efficiency of 80% is needed in order to create distant pair ($F > 0.9$) at rate fast than 10 pairs per second. For reasonable efficiency of 20%, we are able to create distant pair at rate 1 pair per second. Our optimized results are a factor of ten improved over the original protocol.

4. FIDELITY ATTRACTORS IN THE BDCZ PROTOCOL

We consider possible underlying rules for the optimized implementations. One way to illustrate a specific rule for optimized implementation is by studying the optimized implementations of BDCZ protocol for power-of-two distances $L = 2^n L_0$. We focus on the number of consecutively successful pumping steps m_r needed for quantum repeater at level r (i.e. distance $L_r = 2^r L_0$ for simple bipartition), with $r = 0, 1, \cdots, n$.

To isolate and identify the rules from our optimized implementation, we only optimize over the parameters of $\{m_r\}_r$. Specifically, (1) the quality of the local operations are characterized by $\eta = 0.995$ and $p = 0.995$ (Defined in Ref [1,2]); (2) we do not optimize over the pair generation between nearest stations, but instead set $F_0 = 1$, $\tau_e = 1$, and the communication time $t_c = 1/10$; (3) we do not optimize the process of entanglement connection, but instead apply symmetric bipartition rule (that is two *identical* distance-l pairs are connected to create a distance-$2l$ entangled pair, and entangled pair at level-r has distance $L_r = 2^r L_0$). Since $F_0 = 1$, there is no need to pump at the zeroth level, and $m_0 = 0$. Therefore, only n parameters, $\{m_r\}_{r=1,\cdots,n}$, need to be optimized.

Suppose we want to produce entangled pair at level $n = 50$, and want to compare the optimized implementations for different final fidelities. In Fig. 2, we use a color coding scheme to plot optimized implementations. Each row in Fig. 2 represents an optimized

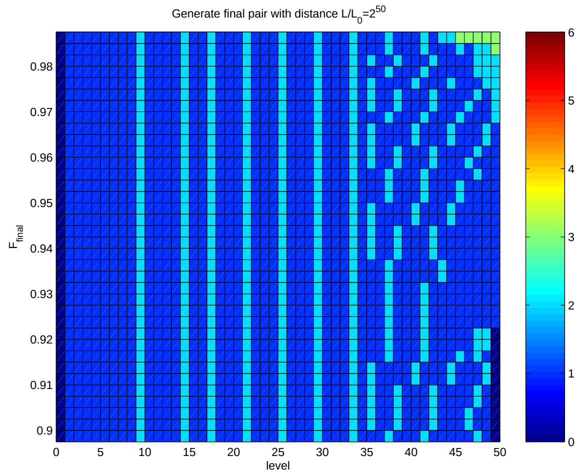

Figure 2: Plaquette plot (F_{final} vs. *level*) of optimized implements for BDCZ protocol with distance $L = 2^{50}L_0$.

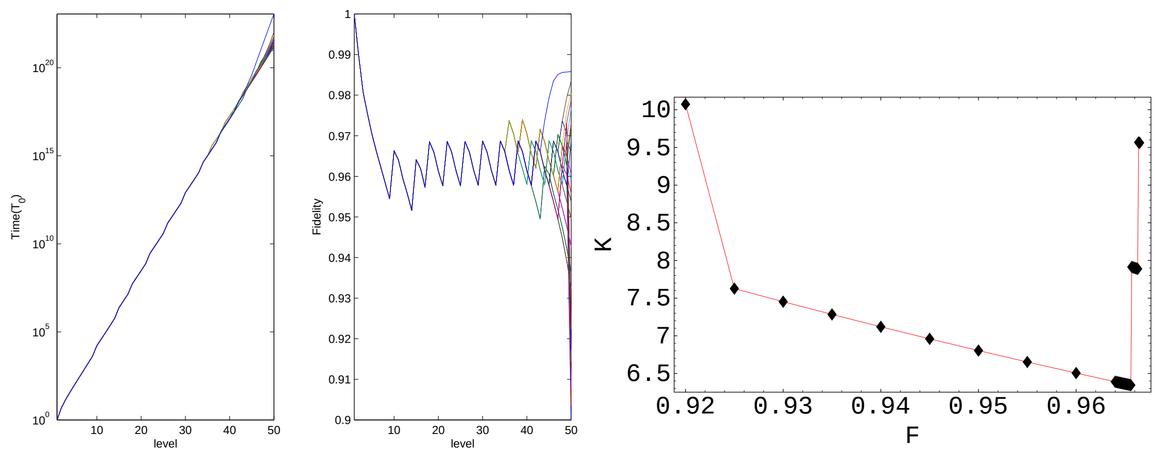

Figure 3: (a) Time scaling for different final fidelities. (b) Fidelity trajectories. The fidelity attractor $F_{attr} \approx 0.965$. (c) Plot of $K(F)$.

protocol to achieve the final fidelity F_{final} for that row. The sequence $\{m_r\}_{r=0,\cdots,n}$ is represented by different colors for each plaquette. Therefore, the number of puming steps needed for rth level in the optimized implementation to create entangled pair with fidelity F_{final} can be readout from the plaquette at the row of F_{final} and the column of r.

There is an obvious pattern of $\{m_r\}$ which is common for a range of values of F_{final}. All the other rows share the same m_r's for $r \leq 35$. For $36 \leq r \leq 44$, the m_r's have similar behavior for neighboring rows. This suggests an underlying rule dictates the choice of m_r.

To understand these patterns, it is helpful to plot the fidelity trajectories (fidelity v.s. level) for the optimal implementations (Fig. 3b). Due to the strong similiarity among these implementations, these fidelity trajectories overlap. Apart from the end of the fidelity trajectories (i.e. in the last few levels), the trajectories all oscillate around some fixed fidelity (e.g. $F_{attr} \approx 0.965$), which we call the *fidelity attractor*.

The value of the fidelity attractor is determined by the balance between the gain in fidelity and the overhead in time. Suppose the fidelity attractor exist, then it should give the minimum time overhead from distance l to distance $2l$. As the creation time grows faster than linear growth with distance, we keep only the leading super-linear parts of Eqs. (2,5)

$$T_A\left(n, F_{A,m}^{(n)}; m, F_B^{(n)}\right) \approx T_B^*\left(n, F_B^{(n)}\right)\left(\sum_{i=1}^{m}\prod_{j=i}^{m}\frac{1}{P_S(j)} + \prod_{j=1}^{m}\frac{1}{P_S(j)}\right) \tag{10}$$

$$T_B^*\left(n, F_B^{(n)}\right) \approx T_A^*\left(\frac{n}{2}, F_A^{(n/2)}\right) \tag{11}$$

where the second equality comes from our choice (in this section) to only optimize over entanglement purification, not connection. The optimal time for generating a type-A pair at distance n is

$$T_A^*\left(n, F_A^{(n)}\right) = \min_{m, F_B^{(n)}} T_A\left(n, F_A^{(n)}; m, F_B^{(n)}\right) = T_A^*\left(\frac{n}{2}, F_A^{(n/2)}\right)\min_{m, F_A^{(n/2)}} K\left(F_A^{(n/2)}, m\right) \tag{12}$$

with

$$K\left(F_A^{(n/2)}, m\right) \equiv \sum_{i=1}^{m}\prod_{j=i}^{m}\frac{1}{P_S(j)} + \prod_{j=1}^{m}\frac{1}{P_S(j)}$$

where $Ps(j)$ depends $F_A^{(n/2)}$ according to Eq.(4), and m depends on both $F_A^{(n/2)}$ and $F_A^{(n)}$.

The existence of fidelity attractor (i.e. $F_A^{(n)} = F_A^{(n/2)} = F_{attr}$) implies

$$T_A^*(n, F_{attr}) = T_A^*\left(\frac{n}{2}, F_{attr}\right)\min_{F_{attr}} K(F_{attr}, m(F_{attr})) \sim \left[\min_{F_{attr}} K(F_{attr})\right]^{\log_2 n} \tag{13}$$

We plot $K(F)$ in Fig. 3c, which gives the minimun at $F_{attr} = 0.966$, consistent with the fidelity attractor obtained from dynamic programming.

5. CONCLUSIONS

We have use computer-aided dynamic programming to optimize the BDCZ quantum repeater protocol. Even for poor detector efficiencies ($\sim 20\%$), we find pair-per-second generation at $1280km$, a factor of 10 improvement over the original scheme. Furthermore, for long distances optimal approaches appear to target a specific fidelity for intermediate distances regardless of the final, desired fidelity.

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